Last Time: Introduction to Linear maps.
w/ many examples

Recall: Let B be a basis of vector space V. Let W be a vector space. Every function $f: B \to W$ extends (linearly) to a linear map

$F: V \to W$ via the formula

$$F\left(\sum_{i=1}^{n} c_i b_i \right) = \sum_{i=1}^{n} c_i f(b_i).$$

Point: Linear maps are determined by where they send a basis of the domain space.

More on Linear Maps

Let $L: V \to W$ be a linear map. The kernel of L is $\ker(L) := \{ v \in V : L(v) = 0_W \}$

The range of L is $\operatorname{ran}(L) := \{ L(v) : v \in V \}$.

NB: $\operatorname{Ker}(L) \subseteq V$ while $\operatorname{ran}(L) \subseteq W$.

Prop: The kernel of L is subspace of $\operatorname{dom}(L)$.

Pf: Let $L: V \to W$ be a linear map. We'll use the subspace test to verify $\operatorname{Ker}(L) \leq V$. Note

$$L(0_V) = L(0 \cdot 0_V) = 0 \cdot L(0_V) = 0_W,$$

so $0_V \in \operatorname{Ker}(L) \neq \emptyset$. Now suppose $u, v \in \operatorname{Ker}(L)$ and $c \in \mathbb{R}$. Now we apply L to $u + cv$:

$L(u+cv) = L(u) + L(cv) = L(u) + cL(v) = 0_W + c \cdot 0_W = 0_W$

Hence $u + cv \in \ker(L)$. Hence, by the subspace Test we have $\ker(L) \leq V$. ▨

Ex: Compute $\ker(L)$ for $L: \mathbb{R}^3 \to \mathbb{R}^2$ w/ $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y+z \end{pmatrix}$.

Sol: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \ker(L)$ iff $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

iff $\begin{pmatrix} x \\ y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

iff $\begin{cases} x = 0 \\ y + z = 0 \end{cases}$

Solving the corresponding linear system: $x = 0$, $y = -z$

$$\therefore \ker(L) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \underline{x = 0}, \ \underline{y = -z} \right\}$$

$$= \left\{ \begin{pmatrix} 0 \\ -z \\ z \end{pmatrix} \in \mathbb{R}^3 : z \in \mathbb{R} \right\}$$

$$= \left\{ t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$ ▨

NB: We computed a basis for $\ker(L)$, namely $\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ ...

Ex: Compute $\ker(L)$ where $L: \mathcal{P}_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is given by $L(c + bx + ax^2) = \begin{pmatrix} 3a - b & 2b + c \\ a - c & a + b + c \end{pmatrix}$

Sol: $c + bx + ax^2 \in \ker(L)$

iff $L(c + bx + ax^2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

iff $\begin{pmatrix} 3a - b & 2b + c \\ a - c & a + b + c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

iff $\begin{cases} 3a - b = 0 \\ 2b + c = 0 \\ a - c = 0 \\ a + b + c = 0 \end{cases}$

Solving this linear system:

$$\begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & -1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \\ 0 = 0 \end{cases}$$

$$\therefore \ \ker(L) = \{0 + 0x + 0x^2\} = \{0\}. \qquad \square$$

Ex: Compute $\ker(L)$ for $L: \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$L\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x+y+z \\ x-y+w \end{pmatrix}.$$

Sol: $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \ker(L)$ iff $L\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

iff $\begin{pmatrix} x+y+z \\ x-y+w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

iff $\begin{cases} x + y + z = 0 \\ x - y + w = 0 \end{cases}$ ← solve!

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \rightsquigarrow \begin{cases} x + \frac{1}{2}z + \frac{1}{2}w = 0 \\ y + \frac{1}{2}z - \frac{1}{2}w = 0 \end{cases}$$

$\therefore \begin{cases} x = -\frac{1}{2}s - \frac{1}{2}t \\ y = -\frac{1}{2}s + \frac{1}{2}t \\ z = s \\ w = t \end{cases}$ $\quad \therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}s - \frac{1}{2}t \\ -\frac{1}{2}s + \frac{1}{2}t \\ 1s + 0t \\ 0s + 1t \end{pmatrix} = s\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$

$$\therefore \ker(L) = \operatorname{span}\left\{ \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \\ 1 \end{pmatrix} \right\}$$ ▨

What about the range space?

Prop: Suppose $L: V \to W$ is a linear map. Then $\operatorname{ran}(L) \leq W$.

pf: Let $L: V \to W$ be a linear map. We apply the Subspace Test. Note $L(0_V) = 0_W \in \operatorname{ran}(L) \neq \emptyset$.

Given $u, v \in \operatorname{ran}(L)$ and $c \in \mathbb{R}$, notice $u = L(\alpha)$ and $v = L(\beta)$ for some $\alpha, \beta \in V$.

Note $u + cv = L(\alpha) + cL(\beta) = L(\alpha) + L(c\beta) = L(\alpha + c\beta)$.

But $\alpha + c\beta \in V$, so $u + cv = L(\alpha + c\beta)$ yields $u + cv \in \operatorname{ran}(L)$. Hence $\operatorname{ran}(L) \leq W$ by the subspace test. ▨

NB: You can adapt this proof to show that, given $L: V \to W$ and $U \leq V$, $L(U) := \{L(u) : u \in U\} \leq W$...

Ex: Compute $\operatorname{ran}(L)$ where $L: \mathcal{P}_2(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ is given by $L(c + bx + ax^2) = \begin{pmatrix} 3a-b & 2b+c \\ a-c & a+b+c \end{pmatrix}$

Sol: $\operatorname{ran}(L) = \{L(v) : v \in V\}$

$$= \left\{ \begin{pmatrix} 3a-b & 2b+c \\ a-c & a+b+c \end{pmatrix} : a,b,c \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 3a & 0 \\ a & a \end{pmatrix} + \begin{pmatrix} -b & 2b \\ 0 & b \end{pmatrix} + \begin{pmatrix} 0 & c \\ -c & c \end{pmatrix} : a,b,c \in \mathbb{R} \right\}$$

$$= \left\{ a\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} + b\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} : a,b,c \in \mathbb{R} \right\}.$$

$$\therefore \ \operatorname{ran}(L) = \operatorname{span}\left\{ \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\}$$

NB: Earlier we showed this set is Lin indep. also :) ☒

Ex: Compute ran(L) for $L : \mathbb{R}^4 \to \mathbb{R}^2$ w/

$$L\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x+y+z \\ x-y+w \end{pmatrix}.$$

Sol: $\operatorname{ran}(L) = \{ L(v) : v \in \mathbb{R}^4 \}$

$$= \left\{ \begin{pmatrix} x+y+z \\ x-y+w \end{pmatrix} : x,y,z,w \in \mathbb{R} \right\}$$

$$= \left\{ x\begin{pmatrix} 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} 0 \\ 1 \end{pmatrix} : x,y,z,w \in \mathbb{R} \right\}$$

$$\therefore \ \operatorname{ran}(L) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

not a basis! (e.g. $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$)

Exercise: Compute a basis from $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ for ran(L). ☒

Up until now: have $\ker(L) \leq V$ and $\operatorname{ran}(L)$

"kernel of L" / "null space of L" — "range space"/"image".

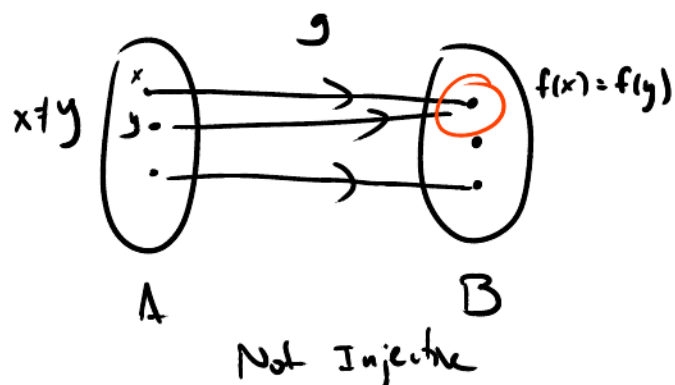
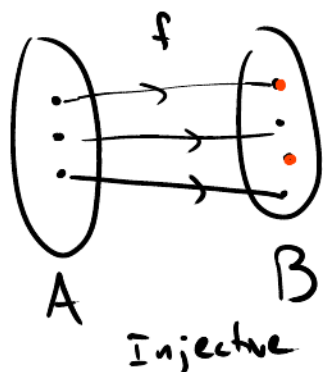
WHY CARE ABOUT THESE SPACES?

Injectivity and Surjectivity

Def^{n}: Let $f: A \to B$ be a function. We say f is <u>injective</u> (or <u>one-to-one</u>) when for all $x, y \in A$, $f(x) = f(y)$ implies $x = y$.

Pictures:

f

A B

Injective

NB: The kernel of a transformation should tell us something about injectivity...

i.e. $\operatorname{Ker}(L) = \{ v \in V : L(v) = 0_W \}$

So if $\ker(L) \neq \{0_V\}$, then $x \in \ker(L)$ w/ $x \neq 0_V$

but $L(x) = 0_W = L(0_V)$

If $\ker(L) \neq \{0_V\}$, then L is not injective.

On the other hand, If L is not injective, then there are $u, v \in V$ w/ $L(u) = L(v)$ but $u \neq v$.

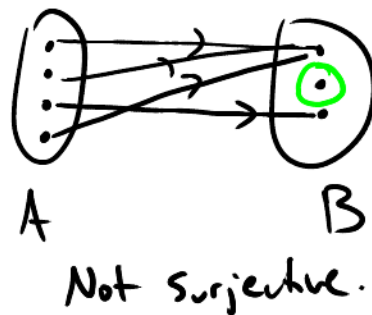
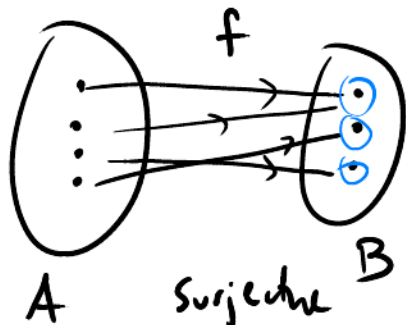
Now $L(u - v) = L(u) - L(v) = 0_W$, but $u \neq v$ implies $u - v \neq 0_V$. Thus, $\ker(L) \neq \{0_V\}$.

Prop: Let $L: V \to W$ be a linear map. L is injective if and only if $\ker(L) = \{0_V\}$. pf: Above :) ☒

Ex: $L(c + bx + ax^2) = \begin{pmatrix} 3a - b & 2b + c \\ a - c & a + b + c \end{pmatrix}$ is injective from earlier work :) ☒

Q: Which of the maps we discussed today were injective?

Defⁿ: A function $f: A \to B$ is surjective (or onto) when for all $b \in B$ there is $a \in A$ w/ $f(a) = b$.

Picture:

Ex: $L\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x+y+z \\ x-y+w \end{pmatrix}$ is surjective.

because $\operatorname{ran}(L) \supseteq \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \mathcal{E}_2$,

($\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $x = y = w = 0$, $z = 1$; $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$: $x = y = z = 0$, $w = 1$)

we see $\mathbb{R}^2 = \operatorname{span}(\mathcal{E}_2) \leq \operatorname{ran}(L) \leq \mathbb{R}^2$. ☑

NB: If $\operatorname{ran}(L) = \operatorname{cod}(L) = W$ (where $L: V \to W$), then L is surjective (by definition). If L is surjective, then $\operatorname{ran}(L) = \{L(v) : v \in V\} = W$ b/c every vector $w \in W$ is $L(v) = w$ for some $v \in V$.

Prop: The linear map $L: V \to W$ is surjective if and only if $\operatorname{ran}(L) = W$.

Q: What if L is both surjective and injective?

↳ "L is bijective" ⇝ L is a "linear isomorphism".